Introduction. Space tethered systems (STS) stabilized by rotation is a quite interesting and promising direction in the field of cosmonautics. Such systems are intended for solving a wide range of scientific and research tasks, in particular, those that cannot be solved effectively with the help of the existing space technologies, for example, transport operations, creation of artificial gravity, removal of space debris objects, obtainment of experimental data of functioning tethered systems, etc.

Problem Statement. The peculiarities of the STS dynamics models are determined by the specifics of the problems solved by such systems actual among which is the researches the effects of the end body dynamics on the system motion.

Purpose. To build a mathematical model of the STS dynamics for considering the general regularities of the system motion and to analyze comprehensively the special features of the end body dynamics.

Materials and Methods. The mathematical model of the STS dynamics has been built based on the methods and principles of theoretical mechanics and space flight dynamics. To study the STS dynamics, methods of the theory of oscillations, analytical and numerical integration of differential motion equations have been used.

Results. The simplest model of the STS dynamics consisting of the material point and the end body connected by a tether is presented for the motion under consideration. The possibility of the appearance of internal resonances and their influence on the stabilization processes in the relative motion of the system has been considered.

Conclusions. The proposed model can apply to analyzing the angular oscillation of the end body relative to the tether attachment point, taking into account the effects of the inertial characteristics of the end body, the tether stiffness and the angular velocity of the proper rotation of the system. Practical problems related to the STS dynamics may include the problems of the stability of the end body orientation, resonance modes in the system motion, as well as the problems in creating the prerequisites for the design of the specific STS.

Keywords: space tethered system, mathematical model, stabilization by rotation, end body, and stabilization processes.
well as analysis and definition of the main patterns of motion will create necessary preconditions for planning and developing definite STS projects. So, in the case of some tasks [5, 6] the creation of STS presupposes compliance with program requirements concerning accuracy of motion orientation of the system end bodies. Earlier [7], a mathematical model of dynamics rotating STS of two end bodies has been proposed. The analysis of dynamics of the considered STS with identical end bodies has shown a possibility of nonlinear resonances causing a significant influence on the dynamics of system end bodies in transient modes of motion. It has been demonstrated that a long-period energy transfer from one body to another takes place in the system [8]. And that is why, one of relevant problems of rotating STS dynamics is to study influence caused by dynamics of end bodies on the process of system motion (in particular, this includes study of interaction between oscillations of STS end bodies and self-rotation of the system).

In this research the simplest model of system dynamics for assessment of influence caused by end body dynamics on motion of the rotating STS has been presented. Simplicity of this model will give an opportunity to carry out a qualitative analysis of the end body motion relative to the tether attachment point, to take into account the influence of end body inertial characteristics and tether stiffness, and to assess the possibility of resonances in the system motion.

One end of the tether (material point A) is assumed to move on unperturbed Keplerian orbit, with the other end of the tether attached to an absolutely rigid body (Fig. 1).

The tether-body system is boosted to have a spin motion around the point A with an angular velocity significantly exceeding that of orbital motion. The connecting tether is sufficiently lightweight and in the research mode of motion it is sufficiently tensioned (i.e. the tether may be viewed as an elastic weightless connection). Energy dissipation of the system motion is performed only by means of internal friction in the elastic tether. The system is assumed to move in a Newtonian force field and there are no other external impacts. It is also presupposed that the system moves only in the orbital plane.

**MOTION EQUATIONS OF THE SYSTEM**

The motion equations of the considered system are as follows:

\[
\begin{align*}
\ddot{\mathbf{R}}_A &= -\frac{\mu}{R^3_A}, \\
m_1\ddot{\mathbf{R}}_1 &= \frac{\mu}{R^3_1} \mathbf{R}_1 \times \mathbf{F}_{tr} - \mathbf{F}_{tr}, \\
\mathbf{L}_1 &= \mathbf{M}_{grav,1} - \mathbf{p}_{1m} \times \mathbf{F}_{tr},
\end{align*}
\]

where \(\mathbf{R}_A\), \(\mathbf{R}_1\) are radius vectors directed outward the Newtonian attracting center towards point A; \(m_1\) is mass of end body 1; \(\mathbf{F}_{tr}\) is tether tension force; \(\mathbf{L}_1\) is kinematics momentum of body motion relative to its own center of mass; \(\mathbf{p}_{1m}\) is radius vector directed from the body center of mass to the point of body attachment to the tether; \(\mathbf{M}_{grav,1}\) is the Newtonian force field momentum acting on the system body; \(\mu\) is the gravitational constant.

The influence caused by tether tension force \(\mathbf{F}_{tr}\) and gravitational momentum of forces \(\mathbf{M}_{grav,1}\) on the body 1 is assumed to be defined by formulas analogous to those presented in [7].

**ACTING FORCE AND MOMENTUM**

The tether elastic properties are described by Hook’s law and the energy dissipation in the tether material is expressed using the formulas of
the equivalent viscous friction:

\[ F_{\nu} = -c \frac{\ddot{r}_i (r_i - d)}{\dot{r}_i} - \chi \dot{r}_i \frac{\ddot{r}_i}{\dot{r}_i} \delta \]

where \( \ddot{r}_i \) is vector directed from the material point A to the attachment point of the end body 1 (attachment to the tether); \( r_i = |r_i|; d \) is nominal length of the tether; \( c \) is stiffness coefficient of the tether; \( \chi \) is damping coefficient [9].

Momentum of gravitational forces acting on the end body of the system is described by the following formula:

\[ M_{grav} = 3 \frac{\mu}{R_i^3} \dot{e}_{R_i} \times J_i \dot{e}_{R_i} \]

where \( J_i \) is inertia tensor of the end body 1; \( \dot{e}_{R_i} \) is unit vector of body relative to point \( R_i \).

Based on (1), the equation of relative motion (the center-of-mass motion equation of the end body relative to point A) has been obtained:

\[ \ddot{r} = \dot{R}_i - \ddot{R}_A = \left[ -\frac{\mu}{R_i^3} \dot{e}_{R_i} + \frac{\mu}{R_A^3} \dot{e}_{R_A} \right] - \dot{F}_n / m_1. \]

Presupposing that the end body has a spherical shape, the equation of end body motion relative to its center of mass can be written as

\[ \ddot{J}_i \omega_i = -\dot{\rho}_{in} \times \dot{F}_n, \]

where \( \omega_i \) is vector of absolutely angular velocity of the end body motion.

In (2), let us expand \( \ddot{r} \) into a \( \left( \frac{r}{R} \right)^3 \) power series neglecting the higher order terms. The symbol \( R = R_A \) is introduced for convenience and simplicity. As far as \( r \) is hundreds meters and \( R \approx 7021 \text{ km} \), the value \( \left( \frac{r}{R} \right)^3 \) has an order of \( 10^{-8} \) and therefore, may be neglected.

Taking into account that \( \frac{1}{R_i^3} = \frac{1}{R^3} \left( 1 - 3 (\dot{e}_{R_i} \cdot \dot{e}_{R_i}) \frac{r}{R} \right) \), the equation for \( \ddot{r} \) may be presented as follows:

\[ \ddot{r} = \frac{\mu}{R^3} \left[ 3 \dot{e}_{R_i} (\dot{e}_{R_i} \cdot \dot{e}_{R_i}) - \ddot{r}_i \right] - \dot{F}_n / m_1, \]

where \( \dot{e}_{R_i} \) is unit vector of the \( \dot{R}_i \); \( \dot{e}_{R_i} \) is unit vector of the \( \ddot{r} \).

**KINEMATICS OF THE SYSTEM**

To study the motion of the considerable system let us introduce right-handed coordinate systems similar to [7], Fig. 4.

\[ OX_u Y_u Z_u \]

is inertial coordinate system (ICS) with the origin in the center of the Earth \( O \). \( OX_u \)

is directed to the vernal equinox; \( OZ_u \)

is directed along the Earth rotational axis;

\[ OX_a Y_a Z_a \]

is orbital coordinate system (OCS) with

the origin in point \( O_a \) coinciding with the center of mass of the system (with point A). \( OX_a \)

is directed along the radius vector connecting the center of mass of the system with the Earth’s center \( O_a Y_a \) in the plane of instantaneous orbit towards the motion of system center of mass;

\[ OX_b Y_b Z_b \]

is moving coordinate system related with the STS body motion plane (CCS) with the origin in the center of mass \( O_b \) of the end body 1;

\( O_x y_z \) is coordinate system related to the end body (BCS) with the origin in the center of mass \( O \) of the end body 1 (Fig. 2). The axis coincides with the principal central axis of inertia of the body.

In accordance with the task set, the system center of mass is assumed to move on the Keplerian orbit, only in the plane of the orbit. In this case, the mutual orientation of coordinate systems can be described as (Fig. 2): \( OX_a Y_a Z_a \) and \( OX_u Y_u Z_u \)

are Euler angles (true anomaly angle \( \nu \)), \( \nu = \omega_u + \delta \), \( \omega_u \) is angular velocity of the center of mass motion on the orbit or \( \nu = \frac{\sqrt{\mu R}}{R} t \); \( OX_a Y_a Z_a \) and \( OX_b Y_b Z_b \) are Euler angles (pure rotation angle \( \phi' \)); \( O_x y_z \) and \( O_x Y_b Z_b \) are Krylow angles (yaw angle \( \psi \)).

**MOTION EQUATIONS OF IN THE SCALAR FORM**

For the purpose of numerical investigations concerning the end body dynamics in the rotational motion of the system let us develop formulas for \( \ddot{r} \) and \( \dot{\omega} \) in the projections on the CCS axis.

Vector \( \ddot{r} \) may be presented as \( \ddot{r} = \dot{r} \dot{\phi}_R \).

Having differentiated \( \ddot{r} \) on time, we receive the formulas for definition of \( \dot{r}, \ddot{r} \).

\[ \ddot{r} = \dot{r} \dot{\phi}_R + r \dot{\phi}_R^2 \]
where $\hat{e}_y$ is unit vector of the axis $Y_c$.

$$\vec{r} = (r - r \dot{\phi}) \hat{e}_r + (r \ddot{\phi} + 2 \dot{r} \dot{\phi}) \hat{e}_y,$$  where $\dot{\phi}$ is angular velocity of CCS relative to ICS ($\omega_{cz} = \dot{\phi}, \varphi = v + \varphi'$).

Let’s write expressions for $F_{tr}$ in the projections on the CCS axis as

$$F_{tr} \beta^{(c)} = \delta \left[ c \left( \frac{\dot{r}_l - \delta}{\delta} + \chi \dot{r}_l \right) \right]_{\beta^{(c)}}, \delta = \begin{cases} 0, & r_l < d, \\ 1, & r_l > d, \end{cases}$$

where $\dot{r} = \left( \dot{r}_l, \dot{r}_t \right)$

Proceeding from the geometry of the system (Fig. 2)

$$\vec{r}_l = \vec{r} + \vec{\rho}_{ln}.$$  Having differentiated $\vec{r}_l$ on time we receive:

$$\dot{\vec{r}}_l = \dot{\vec{r}} + \dot{\vec{\rho}}_{ln},$$

where $\dot{\vec{\rho}}_{ln} = \vec{\omega}_{ln} \times \vec{\rho}_{ln}.$

The orientation of body 1 in CCS can be conveniently provided by means of radius vector $\vec{\rho}_{ln}$ and angle $\alpha$ (Fig. 2).

In accordance with [8], the orientation of radius vector $\vec{\rho}_{ln}^1 = -\vec{\rho}_{ln}^1$ (directed from the point of end body attachment (to the tether) to the center of mass of the end body 1) in the CCS is determined by two angles $\alpha_1, \beta_1$ (Fig. 3):

$\alpha_1$ is angle between $\vec{\rho}_{ln}^1$ and the axis $O_1 X_c$;

$\beta_1$ is angle between the projection of the radius vector $\vec{\rho}_{ln}^1 - P \vec{\rho}_{ln}^1$ on the plane $O_c X_c Y_c$ and the axis $O_c X_c$, respectively.

In this case orientation of $\vec{\rho}_{ln}$ in CCS is defined by means of a column of direction cosines of the unit vector $\vec{\rho}^{(c)}_{ln}$

$$\vec{\rho}^{(c)}_{ln} = \rho_{ln} \vec{\rho}^{(c)}_{ln},$$

where $\vec{\rho}^{(c)}_{ln} = -\vec{e}_r \sin \alpha + \vec{e}_y \cos \alpha.$

In projections on CCS axis the expression for $\vec{\rho}_{ln}$ shall be presented as

$$\vec{\rho}^{(c)}_{ln} = \omega_{ln} \rho_{ln} \vec{\rho}^{(c)}_{ln},$$

where $\vec{\rho}^{(c)}_{y_1}$ is unit vector of the axis $O_1 Y_1$ in projections on CCS axis, $\vec{\rho}^{(c)}_{y_1} = -\vec{\rho}_{ln} \sin \alpha + \vec{\rho}_{ln} \cos \alpha.$

In a similar way, $\vec{r}, \vec{\dot{r}}$ in the projections on CCS axis

$$\vec{r}_l = (\vec{r} - \vec{\rho}_{ln} \cos \alpha) \vec{\rho}_{ln} \sin \alpha \vec{e}_y;$$

$$\vec{r}_l = (\vec{r} - \vec{\rho}_{ln} \sin \alpha - \vec{\rho}_{ln} \cos \alpha \vec{e}_y).$$

The expressions for $r_l$ and $\dot{r}_l$ included into $F_{tr}$ in (6) can be easily received by means of permuting (adding) $\vec{r}_l$ and $\vec{\dot{r}}_l$ to them

$$r_l = \sqrt{r^2 - 2 r \rho_{ln} \cos \alpha},$$

$$\dot{r}_l = \dot{r} (r - \rho_{ln} \sin \alpha - \rho_{ln} \cos \alpha \omega_{ln} - \phi).$$

Momentum of the tether tension force, $\vec{M}_{tr} = -\vec{\rho}_{ln} \times \vec{F}_{tr}$, in the projections on CCS axes is

$$\vec{M}^{(c)}_{tr} = \rho_{ln} r \sin \alpha (\vec{e}_y \times \vec{F}_{tr} \vec{\rho})_{z_1},$$

where $\vec{e}_z$ is unit vector of the axis $O_c Z_c.$
Let us write down the right side of the equation (4) in the projections on CCS axes.

Let us present $\mathbf{e}_R$ in the projections on CCS axes. The transition from OCS to CCS is presented in Fig. 2.

$$
\mathbf{e}_R^{(e)} = \mathbf{e}_r \cos \varphi' - \mathbf{e}_y \sin \varphi'.
$$

Having been transformed, the expression for $\mathbf{r}$ in (6) is presented as

$$
\mathbf{r} = \frac{\mu}{R^3} r \left[ (3 \cos^2 \varphi' - 1) \mathbf{e}_r - \frac{3}{2} \sin 2\varphi' \mathbf{e}_y \right] - \\
- \frac{F_u}{m_1 r_1} \left[ (r - \rho \cos \alpha) \mathbf{e}_r - \rho \sin \alpha \mathbf{e}_y \right], \quad (7)
$$

As a result of substitution of (7) in (5) and projecting the obtained expression on CCS axis we shall receive

$$
r - r \dot{\varphi}^2 = \frac{\mu}{R^3} r (3 \cos^2 \varphi' - 1) - \frac{F_u}{m_1 r_1} (r - \rho \cos \alpha),
$$

$$
r \dot{\varphi} + 2\dot{r} \varphi = - \frac{3}{2} \frac{\mu}{R^3} r \sin 2\varphi' + \frac{F_u}{m_1 r_1} \rho \sin \alpha. \quad (8)
$$

Taking into account that $\dot{\omega}_1 = \omega_1 \dot{e}_z$, and considering the fact that unit vectors $\mathbf{e}_z$ and $\mathbf{e}_z$ CCS and BCS coincide, (3) can be presented as

$$
\dot{\omega}_1 = - \frac{1}{J_z} \frac{r_1 \rho \sin \alpha}{r_i} F_u, \quad (9)
$$

where $\dot{\omega}_1 = \varphi + \alpha$. 

*Fig. 4.* Change of $r$ ($\chi = 0.01$)

*Fig. 5.* Change of angle $\alpha$ ($\chi = 0.01$)

*Fig. 6.* Change of $r$ ($\chi = 0.1$)

*Fig. 7.* Change of angle $\alpha$ ($\chi = 0.1$)
So, taking into account (8), (9) the system of equations (1) has the following form:

\[
\begin{align*}
\dot{r} - r \dot{\phi}^2 & = \frac{\mu}{R^3} r (3\cos^2 \phi' - 1) - \frac{F_{tr}}{m_1r_1} (r - \rho_{ls} \cos \alpha), \\
\dot{\rho} + 2\dot{\phi} & = -\frac{3}{2} \frac{\mu}{R^3} r \sin 2\phi' + \frac{F_{tr}}{m_1r_1} \rho_{ls} \sin \alpha,
\end{align*}
\]

(10)

The presented STS motion equations (10) give an opportunity to obtain a closed system of first order equations with 6 unknown variables.

The possibility of internal resonances in the system motion and their influence on the processes of stabilization of the end body oscillations of rotating STS with energy dissipation of longitudinal oscillations has been considered. In the motion of such system, internal resonances may occur when the following frequencies are commensurate: the orbital motion of the system center of mass, the spin motion of STS around its own center of mass (point A), changing distance between the ends of the tether (longitudinal oscillations), and free (angular) oscillations of the end body.

It is obvious that the possible frequency resonance between the longitudinal oscillations of the tether and the angular oscillations of the end body is of the greatest interest. Conditions of resonant motions between longitudinal oscillations of the tether and angular oscillations of the end body are determined analytically on the basis of expressions presented in [10]. In this case, when the STS moves in the orbital plane, the internal resonance of the cosidered oscillations will be observed near the following values of the system characteristics (see Table).

The stabilization process of angular oscillations of the end body in the resonant commensurability with longitudinal oscillations of the tether has been numerically estimated taking into account the influence of longitudinal oscillations damping. In Figs. 4—7, there are presented schedules of longitudinal oscillations of the tether (change of length — \(r\), m) and schedules of angular oscillations of the end body (change of angle \(\alpha\), grad) in time at \(V = 8\) periods (turns) of STS motion in orbit \(V = \frac{\omega_r}{2\pi} T, T\) is time interval, s).

Based on the obtained results it is possible to make comparative estimations of the influence of damping coefficient of longitudinal oscillations \(\chi\) on the stabilization of angular oscillations of the end body. The estimates have been done for values \(\chi\) equal to 0.01 and 0.1 kg/s. The data values \(\chi\) are chosen because of the following reasons [9]:

- for various structural materials of the tether, the characteristic value is \(\chi \approx 0.01\) kg/s;
- more energy losses are caused by friction between the tangential details of special damping devices — structural damping, for which the characteristic value is \(\chi \approx 0.1\) kg/s.

Figs. 4—7 show that for fast damping of angular oscillations of the end body, it is necessary to ensure their strong connection with longitudinal oscillations of the tether and, accordingly, the intensive transfer of energy to longitudinal oscillations. As can be seen from Figs. 6, 7, the dependence of logarithmic decrement of longitudinal oscillations \(\delta = \frac{\chi}{2T_k}\) on the coefficient of damping \(\chi = 0.1\) kg/s provides the optimum in the rate of damping of the end body oscillations. Under such \(\chi\), the amplitude of longitudinal oscillations and angular oscillations of the end body “radically” decreases in one period of the STS motion in orbit.

Thus, controlling the tensile strength of the tether or using a longitudinal damper in the resonant setting is an effective means of extinguishing angular oscillations of the end body.

### Characteristics of the STS

<table>
<thead>
<tr>
<th>Characteristics of the tether</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal length</td>
<td>(d = 50) m</td>
</tr>
<tr>
<td>Stiffness</td>
<td>(c = 732) H</td>
</tr>
<tr>
<td>Equilibrium length</td>
<td>(r_1 = 50.83) m</td>
</tr>
</tbody>
</table>

| Inertial characteristics of the end body |  |
| Mass | \(m_1 = 24\) kg |
| Coordinates \(\rho_{ls}\) in projections on the axis BCS | \(\rho_{ls} = [2.165\, m; 0; 0]\) |
CONCLUSION
Hence, the simplest mathematical model of dynamics of the rotating STS that consists a material point and an end body connected with an elastic weightless tether has been presented. This model gives an opportunity to carry out analysis of the end body motion relative to the tether attachment point. The possibility of internal resonances in the rotating STS motion and their influence on the processes of stabilization of the end body oscillations with the energy dissipation of longitudinal oscillations has been considered. In the resonant modes, damping of longitudinal oscillations by several orders of magnitude has been shown to reduce the amplitude of longitudinal oscillations and angular oscillations of the end body and, respectively, the duration of the stabilization processes in relative motion of the STS.

REFERENCES
Матеріали й методи. Побудова математичної моделі динаміки КТС базується на методах і принципах теоретичної механіки, методах динаміки космічного польоту. Для дослідження динаміки КТС використано методи теорії коливань, методи аналітичного та чисельного інтегрування диференційних рівнянь руху.

Результати. Наведено найпростішу для досліджуваного руху модель динаміки КТС, що складається з матеріальної точки і кінцевого тіла, з'єднаних ниткою. Розглянуто можливість виникнення внутрішніх резонансів та їх вплив на процеси стабілізації у відносному русі системи.

Висновки. Запропонована модель динаміки КТС дозволяє виконати аналіз кутових коливань кінцевого тіла відносно точок кріплення до нитки з врахуванням впливу інерційних характеристик кінцевого тіла, жорсткості нитки й кутової швидкості власного руху системи. До практичних питань, пов'язаних з цією задачею, динаміки КТС, можна віднести питання стійкості орієнтації кінцевого тіла, питання про резонансні режими в русі системи, а також питання про створення необхідних передумов для проектування конкретних КТС.

Ключові слова: космічна тросова система, математична модель, стабілізація обертанням, кінцеве тіло, процеси стабілізації.

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ВЛИЯНИЕ ДИНАМИКИ КОНЦЕВОГО ТЕЛА НА ПРОЦЕССЫ СТАБИЛИЗАЦИИ
В ОТНОСИТЕЛЬНОМ ДВИЖЕНИИ КОСМИЧЕСКОЙ ТРОСОВОЙ СИСТЕМЫ,
СТАБИЛИЗИРОВАННОЙ ВРАЩЕНИЕМ

Введение. Использование космических тросовых систем (КТС), стабилизированных вращением, является достаточно новым и перспективным направлением в области современной космонаутики. Такие системы предназначены для решения широкого круга научных и исследовательских задач, в частности, тех, которые невозможно или неэффективно решать с помощью имеющихся средств космической техники, например для транспортных операций, создания искусственной гравитации, увода объектов космического мусора, получения экспериментальных данных функционирования тросовых систем и т. д.

Проблематика. Особенности моделей динамики КТС обусловлены спецификой решаемых такими системами задач, актуальными среди которых являются исследования влияния динамики концевого тела на движение системы.

Цель. Построение математической модели динамики КТС, которая позволит рассмотреть общие закономерности движения системы и выполнить анализ особенностей динамики концевого тела.

Материалы и методы. Построение математической модели динамики КТС базируется на методах и принципах теоретической механики, методах динамики космического полета. Для исследования динамики КТС использованы методы теории колебаний, методы аналитического и численного интегрирования дифференциальных уравнений движения.

Результаты. Представлена простейшая для исследуемого движения модель динамики КТС, состоящая из материальной точки и концевого тела, соединенных нитью. Рассмотрена возможность возникновения внутренних резонансов и их влияние на процессы стабилизации в относительном движении системы.

Выводы. Предложенная модель динамики КТС позволяет выполнить анализ угловых колебаний концевого тела относительно точки крепления к нити с учетом влияния инерциальных характеристик концевого тела, жесткости нити и угловой скорости собственного вращения системы. К практическим вопросам, связанным с данной задачей динамики КТС, можно отнести вопросы устойчивости ориентации концевого тела, вопросы о резонансных режимах в движении системы, а также вопросы о создании необходимых предпосылок для проектирования конкретных КТС.

Ключевые слова: космическая тросовая система, математическая модель, стабилизация вращением, концевое тело, процессы стабилизации.